AQA

Please write clearly in block capitals.

Centre number

Candidate number

Surname

Forename(s)

Candidate signature

I declare this is my own work.

A-level
FURTHER MATHEMATICS
Paper 1

Time allowed: 2 hours

Materials
- You must have the AQA Formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page or on blank pages.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
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10	
11	
12	
13	
14	
15	
TOTAL	

J U N 2 1 7 3 6 7 1 0 1

PB/Jun21/E6

Answer **all** questions in the spaces provided.

1 Find

$$\sum_{r=1}^{20}(r^2 - 2r)$$

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

$\sum_{r=1}^{20}(r^2 - 2r) = \sum_{r=1}^{20} r^2 - 2\sum_{r=1}^{20} r$

$= \frac{20(20+1)(40+1)}{6} - 2\left(\frac{20(20+1)}{2}\right)$

$= 2870 - 420$

$= 2450$

Circle your answer.

[1 mark]

2450   2660   5320   43680

2 Given that $z = 1 - 3\text{i}$ is one root of the equation $z^2 + pz + r = 0$, where p and r are real, find the value of r.

Circle your answer.

[1 mark]

-8 -2 6   10

If p and r are real, then the complex conjugate of $z = 1 - 3i$ is also a root of the equation $z^2 + pz + r = 0$.

Roots ; $z = 1 - 3i$, $z = 1 + 3i$

Product of the roots $= r$

$(1 - 3i)(1 + 3i) = r$

$1 - 3i + 3i - 9i^2 = r$

But $i^2 = -1$

$\Rightarrow \quad 1 - 9(-1) = r$

$1 + 9 = r$

$10 = r$

$\therefore \quad r = 10$

Do not write outside the box

3 The curve C has polar equation

$$r^2 \sin 2\theta = 4$$

Find a Cartesian equation for C.

Circle your answer.

[1 mark]

$y = 2x$ $\qquad y = \frac{x}{2}$ $\qquad y = \frac{2}{x}$ $\qquad y = 4x$

Turn over for the next question

$r^2 \sin 2\theta = 4$ Recall $\sin 2\theta = 2\sin\theta\cos\theta$

$r^2(2\sin\theta\cos\theta) = 4$

$\frac{\cancel{2} r^2 \sin\theta\cos\theta}{\cancel{2}} = \frac{4}{2}$

$r^2 \sin\theta\cos\theta = 2$

$x = r\cos\theta \Rightarrow \cos\theta = \frac{x}{r}$

$y = r\sin\theta \Rightarrow \sin\theta = \frac{y}{r}$

$r^2 \sin\theta\cos\theta = 2$

$r^2\left(\frac{x}{r}\right)\left(\frac{y}{r}\right) = 2$

$\cancel{r^2}\left(\frac{xy}{\cancel{r^2}}\right) = 2$

$\frac{\cancel{x}y}{\cancel{x}} = \frac{2}{x}$

$y = \frac{2}{x}$

Do not write outside the box

4 Show that the solutions to the equation

$$3\tanh^2 x - 2\operatorname{sech} x = 2$$

can be expressed in the form

$$x = \pm \ln(a + \sqrt{b})$$

where a and b are integers to be found.

You may use without proof the result $\cosh^{-1} y = \ln(y + \sqrt{y^2 - 1})$

[5 marks]

$3\tanh^2 x - 2\operatorname{sech} x = 2$

$1 - \tanh^2 x = \operatorname{sech}^2 x$

$\Rightarrow \tanh^2 x = 1 - \operatorname{sech}^2 x$

$3(1 - \operatorname{sech}^2 x) - 2\operatorname{sech} x = 2$

$3 - 3\operatorname{sech}^2 x - 2\operatorname{sech} x = 2$

$3 - 3\operatorname{sech}^2 x - 2\operatorname{sech} x - 2 = 0$

$3\operatorname{sech}^2 x + 2\operatorname{sech} x - 1 = 0$

Let $x = \operatorname{sech} x$

$\Rightarrow 3x^2 + 2x - 1 = 0 \quad \text{-------} \quad (i)$

Solving (i) using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 + \sqrt{2^2 - 4(3)(-1)}}{2(3)}$$

$$= \frac{-2 \pm \sqrt{16}}{6}$$

$$= \frac{-2 \pm 4}{6}$$

$\therefore x = \frac{1}{3}$ or $x = -1$

$\Rightarrow \operatorname{sech} x = \frac{1}{3}$ and $\operatorname{sech} x = -1$

Do not write outside the box

But $\operatorname{sech} x > 0 \Rightarrow \operatorname{sech} x \neq -1$

Therefore $\operatorname{sech} x = \frac{1}{3}$

Recall that $\frac{1}{\cosh x} = \operatorname{sech} x$

$$\frac{1}{\cosh x} = \frac{1}{3}$$

$$\Rightarrow \cosh x = 3$$

$$x = \cosh^{-1} 3$$

But $\cosh^{-1} x = \pm \ln\left(x + \sqrt{x^2 - 1}\right)$

$$x = \cosh^{-1} 3 = \pm \ln\left(3 + \sqrt{3^2 - 1}\right)$$

$$= \pm \ln\left(3 + \sqrt{8}\right)$$

$$\therefore x = \pm \ln\left(3 + \sqrt{8}\right)$$

Turn over for the next question

5 The matrix **M** is defined by $\mathbf{M} = \begin{bmatrix} 3 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Prove by induction that $\mathbf{M}^n = \begin{bmatrix} 3^n & 3^n - 1 & -3^n + 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for all integers $n \geq 1$

[5 marks]

We first show that the result is true for $n=1$,

Let $n=1$, then $M^1 = \begin{bmatrix} 3 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so

the result is true for $n=1$.

Assume that the result is true for $n=k$, that is

$$M^k = \begin{matrix} 3^k & 3^k - 1 & -3^k + 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

we then show that it is also true for $n = k+1$

$$M^{k+1} = M^k \cdot M^1 = M^1 \cdot M^k$$

$$\begin{bmatrix} 3 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 3^k - 1 & -3^k + 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \times 3^k + 0 + 0 & 3(3^k - 1) + (2 \times 1) + 0 & 3(-3^k + 1) + 0 + (-2 \times 1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Recall that $3 \times 3^k = 3^{1+k} = 3^{k+1}$

Do not write outside the box

$$\begin{bmatrix} 3^{k+1} & 3^{k+1}-1 & -3^{k+1}+1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the result is true for $n = k+1$

$\Rightarrow$ It is true for $n=1$. If the result is true for $n=k$, then it is true for $n=k+1$. Therefore by induction

$$M^n = \begin{bmatrix} 3^n & 3^n-1 & -3^n+1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is true for all integers $n \geq 1$

$\square$

Turn over for the next question

Do not write outside the box

6 (a) Show that the equation

$$(2z - z^*)^* = z^2$$

has exactly **four** solutions and state these solutions.

[7 marks]

Let z^* be the complex conjugate of z

Let $z = x + iy$ and $z^* = x - iy$

Consider the left hand side:

$(2z - z^*)^*$

$2z - z^* = 2(x+iy) - (x-iy)$

$= 2x + 2iy - x + iy$

$= x + 3iy$

$(2z - z^*)^* = x - 3iy$

$z^2 = (x+iy)^2$

$= x(x+iy) + iy(x+iy)$

$= x^2 + ixy + ixy + yi^2$

$= x^2 + 2ixy - y^2$

$= x^2 - y^2 + 2ixy$

$\Rightarrow (2z - z^*)^* = z^2$

$x - 3iy = x^2 - y^2 + 2ixy$

Equating the real part on the Right hand Side to the real part on the left hand side and also the imaginary part on the Right hand Side and the imaginary part on the left hand side:

Re : $x = x^2 - y^2$ - - - - - - - (i)

Im : $-3y = 2xy$ - - - - - . . (ii)

From (ii) $-3y = 2xy$

$\Rightarrow 2xy + 3y = 0$

$y(2x+3)=0$

$\Rightarrow \quad y=0 \quad$ or $\quad 2x+3=0 \quad \Rightarrow \quad 2x=-3$

$x=-\frac{3}{2}$

$\therefore \quad y=0 \quad , \quad x=-\frac{3}{2}$

When $y=0$, from (1) $\quad x=x^2-y^2$

$\Rightarrow \quad x^2-y^2-x=0$

$x^2-x=0$

$x(x-1)=0$

$x=0 \quad$ or $\quad x=1$

When $x=-3/2$, $\quad \left(\frac{-3}{2}\right)^2-y^2-\left(\frac{-3}{2}\right)=0$

$\frac{9}{4}-y^2+\frac{3}{2}=0$

$y^2=\frac{15}{4} \quad \Rightarrow \quad y=\sqrt{\frac{15}{4}}$

$\therefore \quad y=\pm\frac{\sqrt{15}}{2}$

$z=x+iy$

The only solutions are:

$z=0 \quad , \quad z=1$

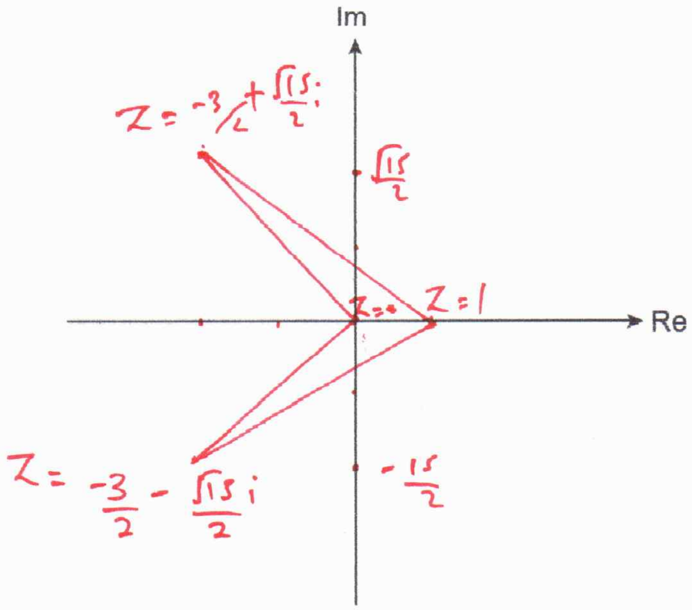
$z=-\frac{3}{2}+\frac{\sqrt{15}}{2}i$ and $z=\frac{-3}{2}-\frac{\sqrt{15}}{2}i$

Hence these are the only solutions and they are exactly four solutions.

Do not write outside the box

6 (b) (i) Plot the four solutions to the equation in part **(a)** on the Argand diagram below and join them together to form a quadrilateral with one line of symmetry.

[2 marks]



6 (b) (ii) Show that the area of this quadrilateral is $\frac{\sqrt{15}}{2}$ square units.

[1 mark]

Area of Upper triangle = Area of Lower triangle

$= \frac{1}{2} \times \frac{\sqrt{15}}{2} \times 1 = \frac{\sqrt{15}}{4}$

Area of the quadrilateral $= 2 \times \frac{\sqrt{15}}{4}$

$= \frac{\sqrt{15}}{2}$ square units.

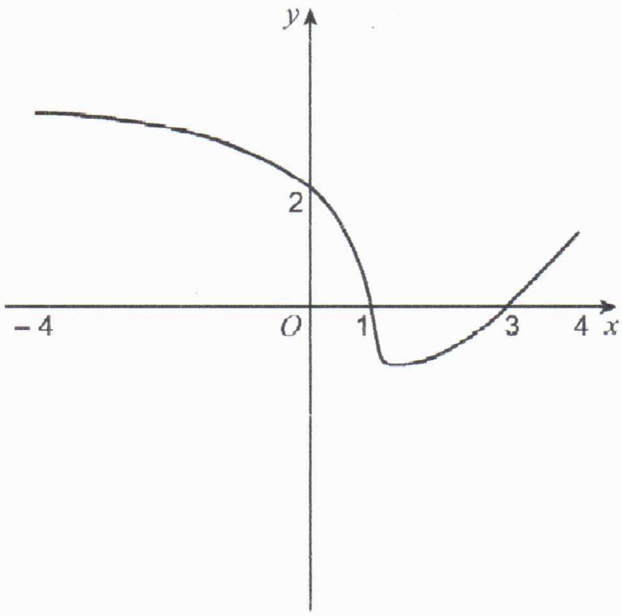
Turn over ▶

Do not write outside the box

7 The diagram below shows the graph of $y = \mathrm{f}(x)$ $(-4 \leq x \leq 4)$

The graph meets the x-axis at $x = 1$ and $x = 3$

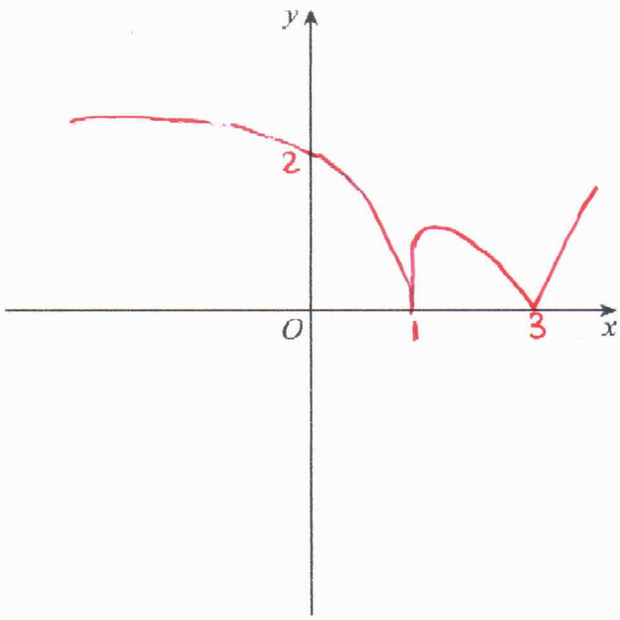
The graph meets the y-axis at $y = 2$



7 (a) Sketch the graph of $y = |\mathrm{f}(x)|$ on the axes below.

Show any axis intercepts.

[2 marks]

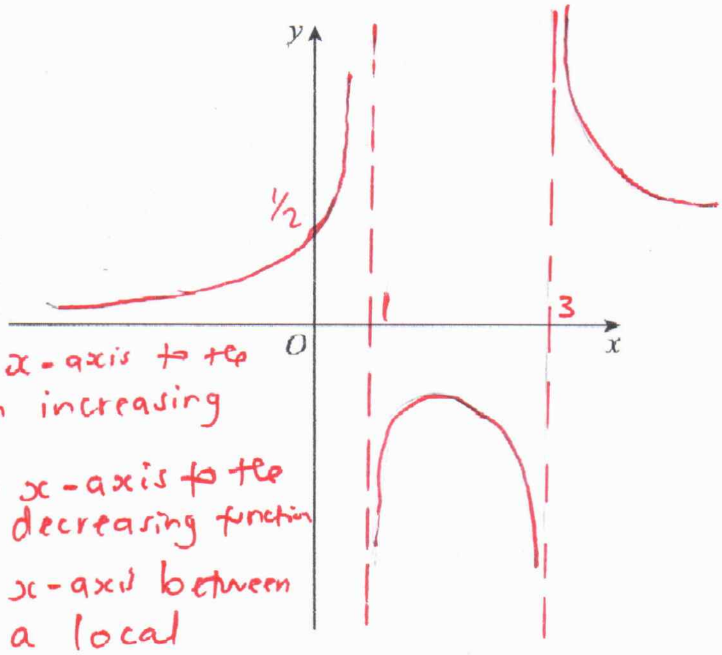


Reflects part of the graph which lies below the x-axis in the x-axis

7 (b) Sketch the graph of $y = \dfrac{1}{f(x)}$ on the axes below.

Show any axis intercepts and asymptotes.

[3 marks]

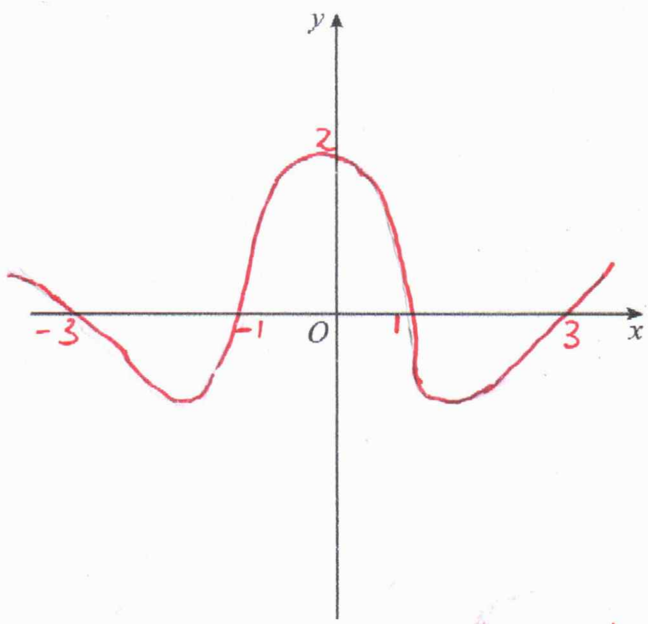


- The curve above the x-axis to the left of $x = 1$ is an increasing function
- The curve above the x-axis to the left of $x = 3$ is a decreasing function
- The curve below the x-axis between $x = 1$ and $x = 3$ has a local maximum.

7 (c) Sketch the graph of $y = f(|x|)$ on the axes below.

Show any axis intercepts.

[2 marks]



Reflects part of the graph which lies to the right of the y-axis in the y-axis.

Reflects graph of $y = f(x)$

8 A particle of mass 4 kg moves horizontally in a straight line.

At time t seconds the velocity of the particle is $v\,\mathrm{m\,s^{-1}}$

The following horizontal forces act on the particle:

- a constant driving force of magnitude 1.8 newtons
- another driving force of magnitude $30\sqrt{t}$ newtons
- a resistive force of magnitude $0.08\,v^2$ newtons

When $t = 70$, $v = 54$

Use Euler's method with a step length of 0.5 to estimate the velocity of the particle after 71 seconds.

Give your answer to **four** significant figures.

[6 marks]

Using newton's second law of motion:

$F_{net} = ma$

$1\cdot 8 + 30\sqrt{t} - 0\cdot 8v^2 = 4a$

But $a = \frac{dv}{dt}$

$1\cdot 8 + 30t^{1/2} - 0\cdot 8v^2 = 4\frac{dv}{dt}$

$\Rightarrow \frac{dv}{dt} = \frac{1}{4}(1\cdot 8 + 30t^{1/2} - 0\cdot 8v^2)$

$\frac{dv}{dt} = 0\cdot 45 + 7\cdot 5t^{1/2} - 0\cdot 02v^2$

Using eulers method

$V_{n+1} = V_n + hv'(t)$

$V_1 = V_0 + h$

$V_2 = V_1 + h$

Do not write outside the box

$V_{70.5} = 54 + 0.5 \left(\dot{V}_{70} \right)$

$= 54 + 0.5 \left(0.45 + 7.5 \left(70^{\frac{1}{2}} \right) - \left(0.02 \left(54^2 \right) \right) \right)$

$= 54 + 0.5 \left(4.8795 \right)$

$= 54 + 2.43975$

$= 56.43975$

$V_{71} = V_{70.5} + 0.5 \left(\dot{V}_{70.5} \right)$

$V_{71} = 56.43975 + 0.5 \left[0.45 + 7.5 \left(70.5^{\frac{1}{2}} \right) - 0.02 \left(56.43975 \right) \right]$

$= 56.43975 + 0.5 \left(-0.285699 \right)$

$= 56.43975 - 0.1428495$

$= 56.2969 \simeq 56.30$

$\therefore$ Velocity after 71 seconds $= 56.30 \; ms^{-1}$

Turn over for the next question

Do not write outside the box

9 Use l'Hôpital's rule to show that

$$\lim_{x \to \infty} (xe^{-x}) = 0$$

Fully justify your answer.

[4 marks]

l'Hopital's rule states that for any two continuous functions $f(x)$ and $g(x)$. If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an indeterminate form, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

where $f'(x)$ is the derivative of $f(x)$ and $g'(x)$ is the derivative of $g(x)$.

Let $f(x) = x$ and $g(x) = e^x$

Then $xe^{-x} = \frac{f(x)}{g(x)}$

$\lim_{x \to \infty} \frac{f(x)}{g(x)}$ is indeterminate, therefore:

$$\lim_{x \to \infty} (xe^{-x}) = \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$$

$f'(x) = \frac{d}{dx}(x) = 1$ $\quad g'(x) = \frac{d}{dx}(e^x)$

$= e^x$

$$\lim_{x \to \infty} (xe^{-x}) = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{1}{e^x}$$

$= 0$ As required

Do not write outside the box

10 Evaluate the improper integral

$$\int_0^8 \ln x \, dx$$

showing the limiting process.

[6 marks]

$$\int_0^8 \ln x \, dx = \lim_{h \to 0} \int_h^8 \ln x \, dx$$

Using integration by parts $\int u \, dv = uv - \int v \, du$

Let $u = \ln x$ $\quad dv = 1$

$du = \frac{1}{x}$ $\quad v = \int 1 \, dx = x$

$$\int_h^8 \ln x \, dx = \lim_{h \to 0} \left[x \ln x \Big|_h^8 - \int x \cdot \frac{1}{x} \, dx \right]$$

$$= \lim_{h \to 0} \left(x \ln x \Big|_h^8 - \int dx \right)$$

$$= \lim_{h \to 0} \left(x \ln x - x \Big|_h^8 \right)$$

$$= \lim_{h \to 0} \left((8 \ln 8 - 8) - (h \ln h - h) \right)$$

As $\lim_{h \to 0} (h \ln h) = 0$

Therefore $\int_0^8 \ln(x) \, dx = 8 \ln(8) - 8$

Turn over ▶

Do not write outside the box

11 The line L_1 has equation $\mathbf{r} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

The line L_2 has equation $\mathbf{r} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

11 (a) Find the acute angle between the lines L_1 and L_2, giving your answer to the nearest 0.1°

[3 marks]

Let θ be the angle between the lines:

$$\cos\theta = \frac{L_1 \cdot L_2}{|L_1|\,|L_2|}$$

$$= \frac{(2,3,-1)\cdot(-2,1,1)}{\sqrt{2^2+3^2+(-1)^2}\sqrt{(-2)^2+1^2+1^2}}$$

$$= \frac{(2\times -2)+(3\times 1)+(-1\times 1)}{\sqrt{14}\sqrt{6}}$$

$$= \frac{-2}{\sqrt{14}\sqrt{6}}$$

$\cos\theta = -0.218217$

$\theta = \cos^{-1}(-0.218217)$

$\theta = 102.604$

Acute angle between the lines: $180 - \theta$

$= 180 - 102.604$

$= 77.3956 \simeq 77.4$

$\approx 77.4°$ (to the nearest 0.1°)

11 (b) The lines L_1 and L_2 lie in the plane Π_1

11 (b) (i) Find the equation of Π_1, giving your answer in the form $\mathbf{r}.\mathbf{n} = d$

[4 marks]

normal $n_1 = L_1 \times L_2$

$$\begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ -2 & 1 & 1 \end{vmatrix}$$

$$i\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} - j\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} + k\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix}$$

$4i + 0j + 8k$

$$n = \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$d = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$d = (2\times 1) + (2\times 0) + (3\times 2)$

$= 8$

$\therefore$ Equation of Π_1

$$= r \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 8$$

Do not write outside the box

11 (b) (ii) Hence find the shortest distance of the plane Π_1 from the origin.

[1 mark]

Shortest distance $= \dfrac{8}{\sqrt{1^2+0^2+2^2}} = \dfrac{8}{\sqrt{5}}$

11 (c) The points $A(4, -1, -1)$, $B(1, 5, -7)$ and $C(3, 4, -8)$ lie in the plane Π_2

Find the angle between the planes Π_1 and Π_2, giving your answer to the nearest 0.1°

[4 marks]

$AB = B - A$

$$\begin{bmatrix}1\\5\\-7\end{bmatrix} - \begin{bmatrix}4\\-1\\-1\end{bmatrix} = \begin{bmatrix}-3\\6\\-6\end{bmatrix} = -3\begin{bmatrix}1\\-2\\2\end{bmatrix}$$

$AC = C - A$

$$\begin{bmatrix}3\\4\\-8\end{bmatrix} - \begin{bmatrix}4\\-1\\-1\end{bmatrix} = \begin{bmatrix}-1\\5\\-7\end{bmatrix}$$

normal $(n_2) = AB \times AC$

$$\begin{vmatrix} i & j & k \\ 1 & -2 & 2 \\ -1 & 5 & -7 \end{vmatrix}$$

$$= i\begin{vmatrix}-2 & 2\\5 & -7\end{vmatrix} - j\begin{vmatrix}1 & 2\\-1 & -7\end{vmatrix} + k\begin{vmatrix}1 & -2\\-1 & 5\end{vmatrix}$$

$= 4i + 5j + 3k$

$n_2 = \begin{pmatrix}4\\5\\3\end{pmatrix}$

Let α be the angle between Π_1 and Π_2

$$\cos\alpha = \frac{n_1 \cdot n_2}{|n_1||n_2|}$$

$$\cos\alpha = \frac{\begin{bmatrix}1\\0\\2\end{bmatrix} \cdot \begin{bmatrix}4\\5\\3\end{bmatrix}}{\sqrt{1^2+0^2+2^2}\sqrt{4^2+5^2+3^2}}$$

$$\cos\alpha = \frac{(1\times 4) + 0 + (2\times 3)}{\sqrt{5}\sqrt{50}}$$

$$\cos\alpha = \frac{10}{\sqrt{5}\sqrt{50}}$$

$\cos\alpha = 0.63246$

$\alpha = \cos^{-1}(0.63246)$

$\alpha = 50.768 \simeq 50.8$ to the nearest 0.1°

$\therefore$ Angle between Π_1 and $\Pi_2 = 50.8°$

Turn over ►

Do not write outside the box

12 The matrix $\mathbf{A} = \begin{bmatrix} 1 & 5 & 3 \\ 4 & -2 & p \\ 8 & 5 & -11 \end{bmatrix}$, where p is a constant.

12 (a) Given that $\mathbf{A}$ is a non-singular matrix, find $\mathbf{A}^{-1}$ in terms of p.

State any restrictions on the value of p.

[6 marks]

$A^{-1} = \frac{1}{\det A} A^T$

$\det A \quad 1\begin{vmatrix} -2 & p \\ 5 & -11 \end{vmatrix} - 5\begin{vmatrix} 4 & p \\ 8 & -11 \end{vmatrix} + 3\begin{vmatrix} 4 & -2 \\ 8 & 5 \end{vmatrix}$

$= 22 - 5p - 5(-44 - 8p) + 3(36)$

$= 350 + 35p$

Matrix of Cofactors

$\begin{bmatrix} \begin{vmatrix} -2 & p \\ 5 & -11 \end{vmatrix} & \begin{vmatrix} 4 & p \\ 8 & -11 \end{vmatrix} & \begin{vmatrix} 4 & -2 \\ 8 & 5 \end{vmatrix} \\ \begin{vmatrix} 5 & 3 \\ 5 & -11 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 8 & -11 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 8 & 5 \end{vmatrix} \\ \begin{vmatrix} 5 & 3 \\ -2 & p \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 4 & p \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 4 & -2 \end{vmatrix} \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$\begin{bmatrix} 22 - 5p & 44 + 8p & 36 \\ 70 & -35 & 35 \\ 5p + 6 & 12 - p & -22 \end{bmatrix}^T = \begin{bmatrix} 22 - 5p & 70 & 5p + 6 \\ 44 + 8p & -35 & 12 - p \\ 36 & 35 & -22 \end{bmatrix}$

$A^{-1} = \frac{1}{350 + 35p} \begin{bmatrix} 22 - 5p & 70 & 5p + 6 \\ 44 + 8p & -35 & 12 - p \\ 36 & 35 & -22 \end{bmatrix}$

For $p \neq -10$

Do not write outside the box

12 (b) The equations below represent three planes.

$$x + 5y + 3z = 5$$
$$4x - 2y + pz = 24$$
$$8x + 5y - 11z = -30$$

12 (b) (i) Find, in terms of p, the coordinates of the point of intersection of the three planes.

[4 marks]

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 5 \\ 24 \\ -30 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{350 + 35p} \begin{bmatrix} 22-5p & 70 & 5p+6 \\ 44+8p & -35 & 12-p \\ 36 & 35 & -22 \end{bmatrix} \begin{bmatrix} 5 \\ 24 \\ -30 \end{bmatrix}$$

$$= \frac{1}{350+35p} \begin{bmatrix} 5(22-5p) + 24(70) - 30(5p+6) \\ 5(44+8p) - 35(24) - 30(12-p) \\ 5(36) + 24(35) - 22(-30) \end{bmatrix}$$

$$\frac{1}{350+35p} \begin{bmatrix} 110 - 25p + 1680 - 150p - 180 \\ 220 + 40p - 840 - 360 + 30p \\ 180 + 840 + 660 \end{bmatrix}$$

$$\frac{1}{350+35p} \begin{bmatrix} 1610 - 175p \\ -980 + 70p \\ 1680 \end{bmatrix} = \frac{\cancel{35}}{\cancel{35}(10+p)} \begin{bmatrix} 46 - 5p \\ -28 + 2p \\ 48 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10+p} \begin{bmatrix} 46-5p \\ -28+2p \\ 48 \end{bmatrix}$$

$$\Rightarrow x = \frac{46-5p}{10+p}, \quad y = \frac{-28+2p}{10+p}, \quad z = \frac{48}{10+p}$$

Therefore the point of intersection is :

$$\left(\frac{46-5p}{10+p}, \frac{-28+2p}{10+p}, \frac{48}{10+p} \right)$$

Turn over ▶

Do not write outside the box

12 (b) (ii) In the case where $p = 2$, show that the planes are mutually perpendicular.

[4 marks]

When $p = 2$

$x + 5y + 3z = 5$

$4x - 2y + 2z = 24$

$8x + 5y - 10z = -30$

$$\begin{bmatrix}1\\5\\3\end{bmatrix} \cdot \begin{bmatrix}4\\-2\\2\end{bmatrix} = (4\times1) + (5\times-2) + (3\times2)$$

$$= 4 - 10 + 6$$

$$= 0$$

Therefore the planes represented by the first two equations are perpendicular because their dot product is equal to zero.

$$\begin{bmatrix}1\\5\\3\end{bmatrix} \times \begin{bmatrix}4\\-2\\2\end{bmatrix} = \begin{vmatrix} i & j & k \\ 1 & 5 & 3 \\ 4 & -2 & 2 \end{vmatrix}$$

$$= i\begin{vmatrix}5 & 3\\-2 & 2\end{vmatrix} - j\begin{vmatrix}1 & 3\\4 & 2\end{vmatrix} + k\begin{vmatrix}1 & 5\\4 & -2\end{vmatrix}$$

$$= 16i + 10j - 22k$$

$$= \begin{bmatrix}16\\10\\-22\end{bmatrix} = 2\begin{bmatrix}8\\5\\-11\end{bmatrix}$$ which is a multiple of the normal vector of the third plane.

$\Rightarrow$ Because the vector perpendicular to the first two planes is a multiple of the normal vector of the third plane, the three planes are mutually perpendicular.

13 The transformation S is represented by the matrix $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

The transformation T is a translation by the vector $\begin{bmatrix} 0 \\ -5 \end{bmatrix}$

Kamla transforms the graphs of various functions by applying first S, then T.

Leo says that, for some graphs, Kamla would get a different result if she applied first T, then S.

Kamla disagrees.

State who is correct.

Fully justify your answer.

[3 marks]

Transformation S then T

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 3x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 3x \\ y-5 \end{bmatrix}$$

Transformation T then S

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -5 \end{bmatrix} = \begin{bmatrix} x \\ y-5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y-5 \end{bmatrix} = \begin{bmatrix} 3x \\ y-5 \end{bmatrix}$$

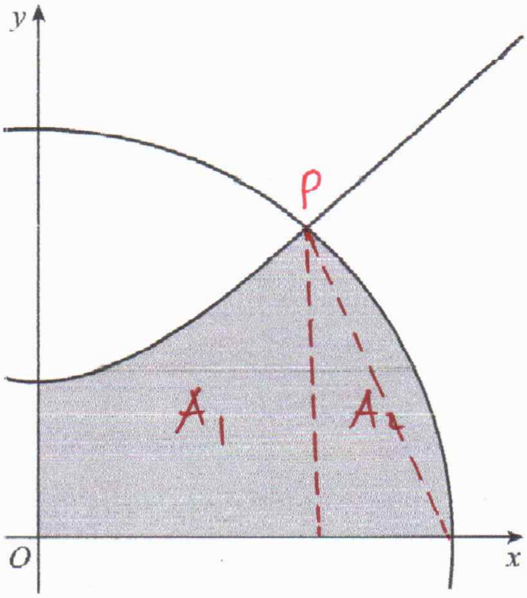
The two transformations are the same, therefore Kamla is correct.

Do not write outside the box

14 The hyperbola H has equation $y^2 - x^2 = 16$

The circle C has equation $x^2 + y^2 = 32$

The diagram below shows part of the graph of H and part of the graph of C.



Show that the shaded region in the first quadrant enclosed by H, C, the x-axis and the y-axis has area

$$\frac{16\pi}{3} + 8\ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)$$

[12 marks]

Let the point of intersection of H and C : P

$\Rightarrow$ $y^2 - x^2 - 16 = x^2 + y^2 - 32$

$\cancel{y^2} - x^2 - 16 - x^2 - \cancel{y^2} + 32 = 0$

$-2x^2 + 16 = 0$

$x^2 = 8$

$\Rightarrow$ $x = \sqrt{8} = 2\sqrt{2}$

At P, $x = 2\sqrt{2}$

Area A_1 $\int_0^{2\sqrt{2}} y \, dx$

$y^2 - x^2 = 16$

$\Rightarrow$ $y^2 = 16 + x^2$

$y = (x^2 + 16)^{1/2}$

$$A_1 = \int_0^{2\sqrt{2}} (x^2+16)^{\frac{1}{2}}\,dx$$

Let $x = 4\sinh u$

$dx = 4\cosh u\,du$

$\therefore (x^2+16)^{\frac{1}{2}} = ((4\sinh u)^2+16)^{\frac{1}{2}} = (16\sinh^2 u + 16)^{\frac{1}{2}}$ [Recall $\cosh^2 u - \sinh^2 u = 1$]

$= (16\cosh^2 u)^{\frac{1}{2}} = 4\cosh u$

$$A_1 = \int_0^{2\sqrt{2}} 4\cosh u \cdot 4\cosh u\,du$$

$$= \int_0^{2\sqrt{2}} 16\cosh^2 u\,du$$

$$= \int_0^{2\sqrt{2}} 16\left(\frac{1}{2}(\cosh 2u + 1)\right) du$$

$$= 8\int_0^{2\sqrt{2}} \cosh 2u + 1\,du$$

$$= 8\left[\frac{\sinh 2u}{2} + u\right]_0^{2\sqrt{2}}$$

$$= \left[4\sinh 2u + 8u\right]_0^{2\sqrt{2}}$$

When $x = 2\sqrt{2}$

$\sinh u = \frac{x}{4} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$

$\cosh u = \frac{(x^2+16)^{\frac{1}{2}}}{4} = \frac{((2\sqrt{2})^2+16)^{\frac{1}{2}}}{4}$

$= \frac{\sqrt{6}}{2}$

But $\sinh 2u = 2\sinh u\cosh u$

$= 2 \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{6}}{2}$

$= \sqrt{3}$

$u = \sinh^{-1}(x)$

$\sinh^{-1} x = \ln x + \sqrt{x^2+1}$

$\sinh^{-1}(2\sqrt{2}) = \ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)$

$\therefore A_1 = 4\sqrt{3} + 8\ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)$

OP makes an angle of $\pi/3$ with the x-axis

Area of the sector $= \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 32 \times \frac{\pi}{3} = \frac{16\pi}{3}$

Area of the triangle $= \frac{1}{2} \times 2\sqrt{2} \times 2\sqrt{6}$

$= 4\sqrt{3}$

$A_2 = \frac{16\pi}{3} - 4\sqrt{3}$

Required area $= A_1 + A_2$

$= \frac{16\pi}{3} + 8\ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right) + \cancel{4\sqrt{3}} - \cancel{4\sqrt{3}}$

$\therefore$ Area

$= \frac{16\pi}{3} + 8\ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)$ As required

Do not write outside the box

15 **In this question use $g = 9.8\,\text{m s}^{-2}$**

A particle P of mass m is attached to two light elastic strings, AP and BP.

The other ends of the strings, A and B, are attached to fixed points which are 4 metres apart on a rough horizontal surface at the bottom of a container.

The coefficient of friction between P and the surface is 0.68

- When the extension of string AP is e_A metres, the tension in AP is $24me_A$
- When the extension of string BP is e_B metres, the tension in BP is $10me_B$
- The natural length of string AP is 1 metre
- The natural length of string BP is 1.3 metres

15 (a) Show that when $AP = 1.5$ metres, the tension in AP is equal to the tension in BP.
[1 mark]

tension in Ap = $24me_A$

$e_A = (1.5 - 1) = 0.5$

$\therefore$ tension in AP $= 24m(0.5)$

$= 12m$

tension in BP $= 10me_B$

$e_B = 4 - (1.3 + 1.5)$

$4 - 2.8$

$= 1.2$

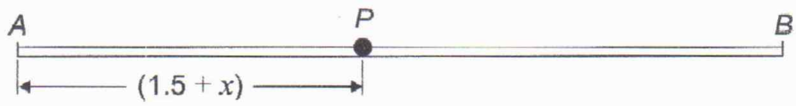
$\therefore$ tension in BP $= 10m(1.2)$

$= 12m$

Therefore the tension in AP is equal to the tension in BP.

15 (b) *P* is held at the point between *A* and *B* where *AP* = 1.9 metres, and then released from rest.

At time t seconds after *P* is released, *AP* = $(1.5 + x)$ metres.



Show that when *P* is moving towards *A*,

$$\frac{d^2x}{dt^2} + 34x = 6.664$$

[3 marks]

Using newton's second law of motion

$F_{net} = ma$ where $a = \frac{d^2x}{dt^2}$

$\mu = 0.68$
$R = mg$

$10m(1.2 - x) - 24m(0.5 + x) + \mu R = m\frac{d^2x}{dt^2}$

$\left(12\cancel{m} - 10mx - 12\cancel{m} - 24mx + 0.68 \times m \times 9.8\right) = m\frac{d^2x}{dt^2}$

$-34mx + 6.664m = m\frac{d^2x}{dt^2}$

$\cancel{m}(-34x + 6.664) = \cancel{m}\frac{d^2x}{dt^2}$

$\frac{d^2x}{dt^2} = -34x + 6.664$

Question 15 continues on the next page

$\therefore \frac{d^2x}{dt^2} + 34x = 6.664$ As required.

15 (c) The container is then filled with oil, and P is again released from rest at the point between A and B where $AP = 1.9$ metres.

At time t seconds after P is released, the oil causes a resistive force of magnitude $10mv$ newtons to act on the particle, where $v\,\text{m s}^{-1}$ is the speed of the particle.

Find x in terms of t when P is moving towards A.

[9 marks]

$m\frac{d^2x}{dt^2} = 10mv + 6.664m - 34mx$

$\frac{d^2x}{dt^2} = 10v + 6.664 - 34x$

But $v = -\frac{dx}{dt}$

$\frac{d^2x}{dt^2} = 10\left(-\frac{dx}{dt}\right) + 6.664 - 34x$

$\therefore \frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 34x = 6.664 \ldots (i)$

Solving equation (i) Complementary function:

Let $\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 34x = 0$

$m^2 + 10m + 34 = 0$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$m = \frac{-10 \pm \sqrt{10^2 - 4(1)(34)}}{2}$

$= \frac{-10 \pm \sqrt{-36}}{2}$

$= \frac{-10 \pm 6i}{2}$

$\therefore m = -5 \pm 3i$

$\therefore$ CF is:

$x = Ae^{-5t}\cos 3t + Be^{-5t}\sin 3t$

Particular Integral PI:

Let $x = \lambda$

$\frac{dx}{dt} = 0, \quad \frac{d^2x}{dt^2} = 0$

Replacing the values in (i)

$0 + 0 + 34(\lambda) = 6.664$

$\lambda = \frac{6.664}{34}$

$\lambda = 0.196$

General solution

$x = Ae^{-5t}\cos 3t + Be^{-5t}\sin 3t + 0.196$

when $t = 0$, $x = 0.4$

$0.4 = A + 0.196$

$\therefore A = 0.4 - 0.196 = 0.204$

$\frac{dx}{dt} = \left(-5Ae^{-5t}\cos 3t - 3Ae^{-5t}\sin 3t - 5Be^{-5t}\sin 3t + 3Be^{-5t}\cos 3t\right)$

when $t = 0$, $\frac{dx}{dt} = 0$

$0 = -5A + 3B$

$0 = -5(0.204) + 3B$

$B = \frac{1.02}{3} = 0.34$

$\therefore x = \left(0.204e^{-5t}\cos 3t + 0.34e^{-5t}\sin 3t + 0.196\right)$

END OF QUESTIONS